System Response via Laplace Transform

First Order System

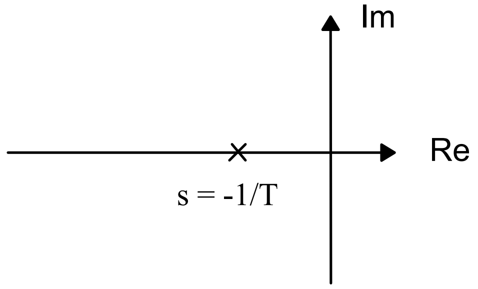
Transfer function: $\frac{x(s)}{u(s)} = \frac{\lambda}{1+sT}$ Differential Equation: $T\frac{dx(t)}{dt} + x(t) = \lambda u(t)$

T is the system's *time constant* [units of time]
λ is the gain [units of x]/[units of u]

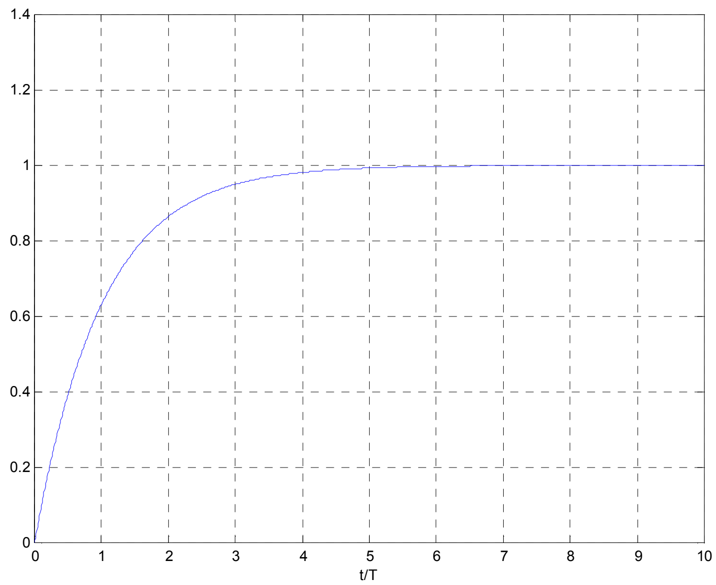
(To save clutter, we assume, without loss of generality, that λ is 1).

Response to unit step: $u(s) = \frac{1}{s}$

System response: $x(s) = \frac{1}{s(1+sT)} = \frac{1}{s} - \frac{T}{1+sT} = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$



$$\Rightarrow \quad x(t) = 1 - e^{-t/T}$$



Step Response of First Order System

x(T) = 63% final value
Settling Time Ts = 4T. i.e. x(Ts) = 98% of final value.
Tangent at t = 0 intercepts final value at t = T
Note that the greater T, the slower the response and the closer s = -1/T to the origin.

Simple 2nd order system

Transfer function: $\frac{x(s)}{u(s)} = \frac{\lambda\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, Differential Eqn: $\frac{d^2x(t)}{dt^2} + 2\zeta\omega_n \frac{dx(t)}{dt} + \omega_n^2 x(t) = \lambda\omega_n^2 u(t)$

ζ is the damping ratio [non-dimensional]

ω_n (>0) is the undamped natural frequency, or bandwidth [rad/unit time]

λ is the gain [units of x]/[units of u]

(We again assume λ is 1).

Unit step input: $u(s) = \frac{1}{s}$

System response: $x(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Case 1: Overdamped (ζ > 1)

$m = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$

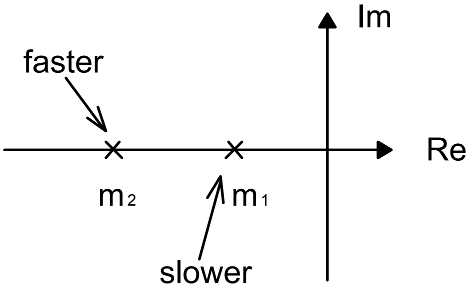
i.e. distinct real negative roots $m_2 < m_1 < 0$

$m_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$

$m_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

$\frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \equiv \frac{A}{s - m_1} + \frac{B}{s - m_2}$ where $A = \frac{\zeta + \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}}$, $B = \frac{-\zeta + \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}}$

System Response: $x(t) = H(t) - Ae^{m_1 t} - Be^{m_2 t}$



For large damping ratio, m_1 tends to zero and m_2 tends to minus infinity, *A* tends to one and *B* tends to zero. The slow root m_1 then dominates the response, which resembles a first-order response with time constant $T = -1/m_1$.

Case 2: Critically damped (ζ = 1)

Roots are equal: $m_1 = m_2 = -\omega_n$

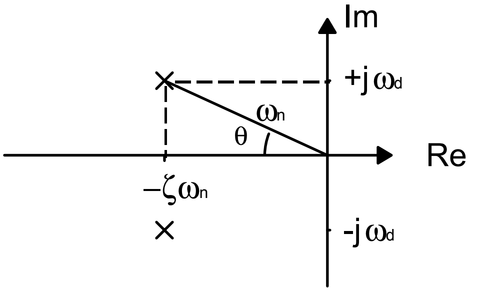
$$x(s) = \frac{1}{s} - \frac{s + 2\omega_n}{(s + \omega_n)^2}$$

$$= \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}$$

System response: $x(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t} = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$

Case 3: Underdamped (ζ < 1)

Roots are complex: $m = -\zeta\omega_n \pm j\omega_d$ where the *damped natural frequency* $\omega_d := \sqrt{1-\zeta^2}\,\omega_n$.
$\cos\theta = \zeta$

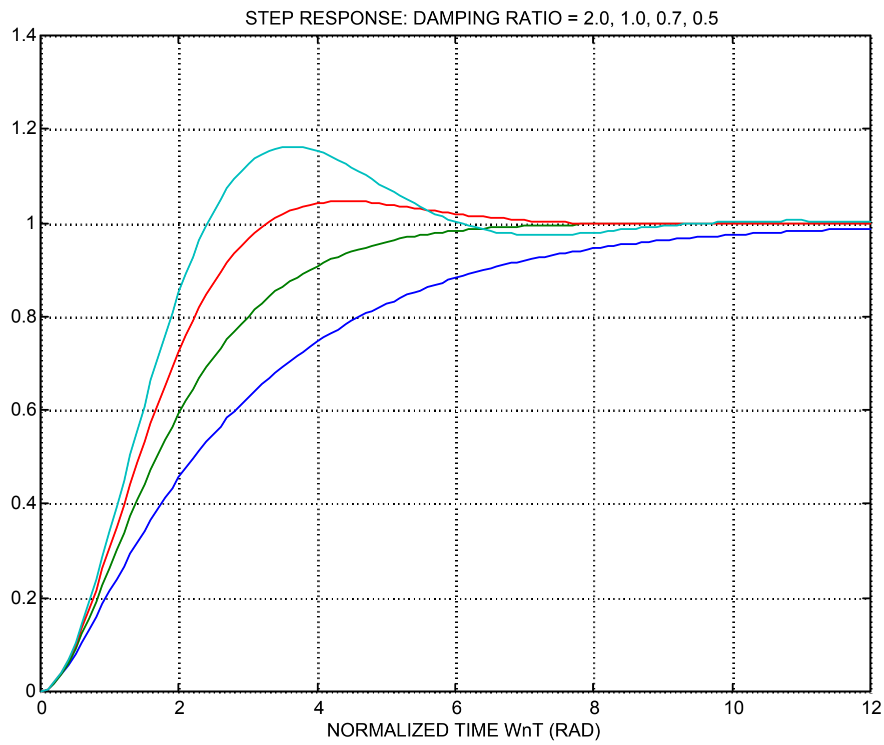


$$x(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{1}{s} - \frac{(s+\zeta\omega_n)+\omega_d(\zeta\omega_n/\omega_d)}{(s+\zeta\omega_n)^2+\omega_d^2}$$

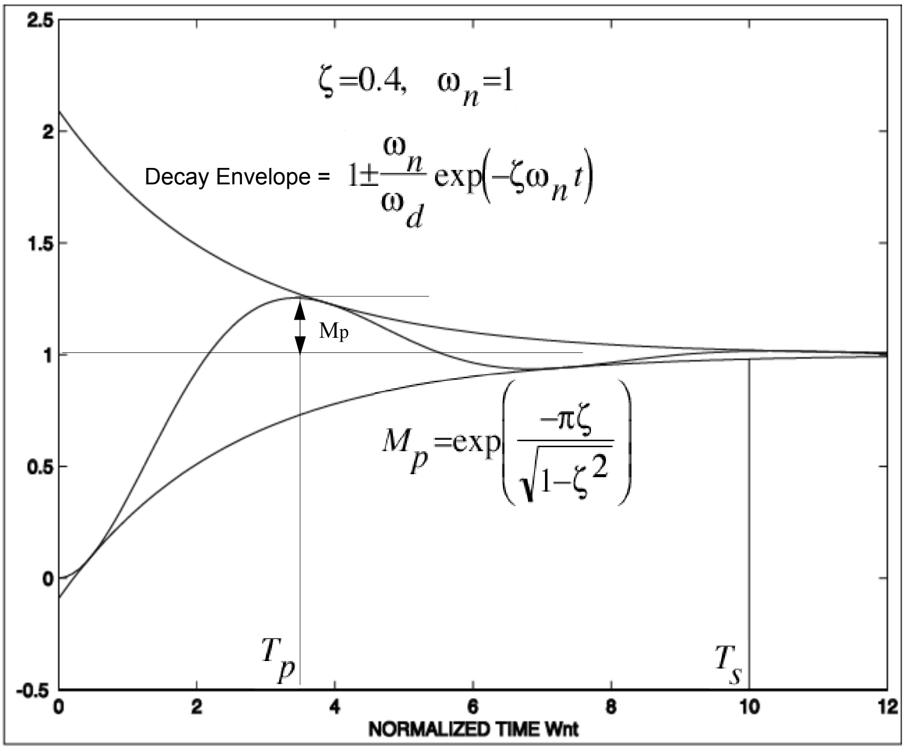
Now $\dfrac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_d^2} \leftrightarrow e^{-\zeta\omega_n t}\cos\omega_d t$ and $\dfrac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2} \leftrightarrow e^{-\zeta\omega_n t}\sin\omega_d t$

$$\Rightarrow x(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)$$

The response is an oscillation at a frequency ω_d that is attenuated exponentially by the factor $e^{-\zeta\omega_n t}$.



Step Responses for overdamped, critically damped and underdamped systems



Step response of underdamped system

Period of oscillation $$P = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Time to first peak $$T_p = 0.5P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Time constant $$T = \frac{1}{\zeta\omega_n}$$

Settling Time $$T_s = 4T = \frac{4}{\zeta\omega_n}$$

Maximum overshoot $$M_p = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

As ω_n increases, so too does ω_d, hence the frequency of oscillations

As ζ increases, number of overshoots decreases, as does magnitude of overshoots

ζ	Mp	Overshoots
0.7	5%	1
0.5	16%	2
0.3	37%	3

Critical damping ($\zeta = 1$) gives the fastest response without overshoot.
Damping of $\zeta = 0.7$ gives a faster response with 5% overshoot, which is sometimes acceptable.

DJW
Feb. 2004